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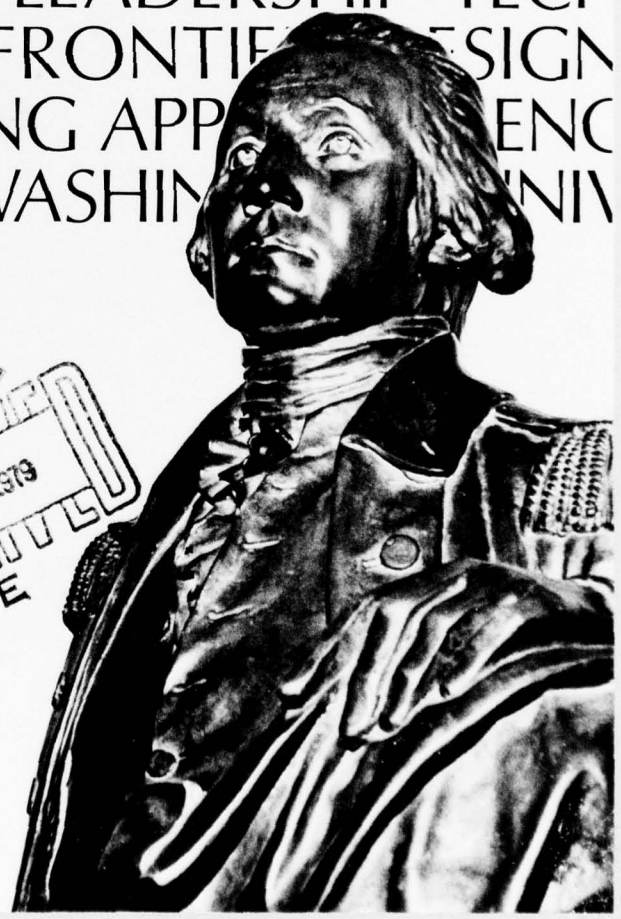
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BASIC RESULTS IN THE DEVELOPMENT OF SENSITIVITY AND STABILITY
ANALYSIS IN CONSTRAINED MATHEMATICAL PROGRAMMING

by

Anthony V. Fiacco
William P. Hutzler

Serial T-407
7 August 1979

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BASIC RESULTS IN THE DEVELOPMENT OF SENSITIVITY AND STABILITY ANALYSIS IN CONSTRAINED MATHEMATICAL PROGRAMMING

by

Anthony V. Fiacco *
William P. Hutzler

Abstract

For large classes of mathematical programming problems, a detailed technical survey is given of key developments in sensitivity and stability analysis results, i.e., results characterizing the relationship between the optimal value function or a solution set and problem perturbations. The emphasis is on finite dimensional nonlinear problems with deterministic parametric perturbations. Precise assumptions and conclusions of key results are given in the more than 30 theorems that are stated. Some effort has been made to unify the notation and terminology and to place the results in perspective. Directions of future research and applications are indicated. Finally, an extensive bibliography is included. The paper is motivated by a desire to unify into one body of theory the many penetrating results that are now known in this crucially important area.

*The Rand Corporation, Washington, D.C.

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CONTENTS

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BASIC RESULTS IN THE DEVELOPMENT OF SENSITIVITY AND STABILITY
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1. Introduction

Many algorithms have been developed, mainly in the last two
decades, for the solution of mathematical programming problems.
However, following the now well known parametric analysis in linear
programming [12,81,95,113] and a fairly comprehensive treatment of the
quadratic parametric problem [17,22,60], until very recently, there
has been only sporadic activity in the study of the solution
sensitivity of general nonlinear mathematical programs to
perturbations of the problem parameters. This paper gives a concise
survey of the state of the art in that area.

As the next two examples demonstrate, the solution of very simple
mathematical programs may not vary at all or may change drastically
for arbitrarily small perturbations of the problem parameters.

Example 1.

Consider the nonlinear program:

$$\min (x_1 - \varepsilon)^2 + (x_2 + 1)^2$$
$$\text{s.t. } x_2 \geq x_1$$
$$x_2 \geq -x_1$$

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The analytical solution of this problem is easily seen to be:

$$x(\varepsilon) = (x_1(\varepsilon), x_2(\varepsilon)) = \begin{cases} ((1+\varepsilon)/2, -(1+\varepsilon)/2) & \varepsilon < -1 \\ (0,0) & -1 \leq \varepsilon \leq 1 \\ ((\varepsilon-1)/2, (\varepsilon-1)/2) & \varepsilon > 1 \end{cases}$$

It is clear that $x(\varepsilon)$ is piecewise linear, continuous, and differentiable everywhere except for $\varepsilon = \pm 1$. It is readily shown that the optimal value function of this problem, $f*(\varepsilon) = f[x(\varepsilon), \varepsilon]$, is twice differentiable everywhere except for $\varepsilon = \pm 1$, where it is only once differentiable.

Unfortunately, as the next example illustrates, the solutions of mathematical programs do not always behave so nicely.

Example 2.

$$\min \ \varepsilon x_1$$
$$\text{s.t.} \ -1 \leq x_1$$

The solution of this problem is given by $x*_1 = -1$ if $\varepsilon > 0$; $x*_1$ can be chosen as any value in $[-1, \infty)$ if $\varepsilon = 0$; and if $\varepsilon < 0$, there is no finite solution of this problem. Thus, as ε varies in a small neighborhood of the origin in E^1, the solution may be finite and unique, may be unbounded, or there may be infinitely many solutions.

It should be clear from these two simple examples that very small perturbations of the parameters of a mathematical program can cause a wide variety of results. The purpose of this paper is to summarize and illustrate the work that has been done to date in 1) providing conditions under which the solutions of nonlinear programs are locally well-behaved, and 2) estimating solution properties as a function of problem parameters.

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2. Objective Function and Constraint Set Continuity

Some of the earliest work in sensitivity analysis for nonlinear programming was concerned with the variation of the optimal objective function value with changes in a parameter appearing in the right-hand side of the constraints, i.e. involving problems of the form:

$$\text{minimize } f(x), \text{ subject to } g(x) \geq \epsilon,$$

where $f: E^n \to E^1$, and $g: E^n \to E^m$. The theory of point-to-set maps (cf. Berge [13]) has been used for much of the analysis of this problem. Hogan [67] has provided an excellent development of those properties of point-to-set maps which are especially useful in deriving such results.

In the next sub-section we present several definitions and properties relating to point-to-set maps which are needed in a number of important results.

2.1 Point-to-Set Maps

Given two sets X and Y, a point-to-set mapping, ϕ, from X to Y is a function which associates with every point in X a subset of Y. Following Berge [13], we say that the point-to-set mapping ϕ is continuous if it is both upper semi-continuous (usc) and lower semi-continuous (lsc) in X. These last two notions are established by the following definitions.

Definition 2.1. Let ϕ be a point-to-set mapping from X to subsets of Y.

    i) ϕ is lsc at $x_o \in X$ if, for each open set $S \subseteq Y$ satisfying $S \cap \phi(x_o) \neq \emptyset$, there exists a neighborhood N of x_o, $N(x_o)$, such that for each x in $N(x_o)$, $\phi(x) \cap S \neq \emptyset$.

    ii) ϕ is usc at $x_o \in X$ if, for each open set $S \subseteq Y$ containing $\phi(x_o)$, there exists a neighborhood N of x_o, $N(x_o)$, such that for each x in $N(x_o)$, $\phi(x) \subset S$.

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An alternative definition of semi-continuity for point-to-set maps is based on the convergence of sets in the range space Y. Let $\{x_n\}$ be any sequence of points in X such that $x_n \to \bar{x}$, and let ϕ be a point- to-set mapping from X to subsets of Y.

Definition 2.2. i) ϕ is lsc (or open) at $\bar{x} \epsilon X$ if, for each $\bar{y} \epsilon \phi(\bar{x})$, there exists a value m and a sequence $\{y_n\} \subseteq Y$ such that $y_n \epsilon \phi(x_n)$ for each $n > m$ and $y_n \to \bar{y}$.
      ii) ϕ is usc (or closed) at $\bar{x} \epsilon X$ if $y_n \epsilon \phi(x)$ and $y_n \to \bar{y}$ together imply that $\bar{y} \epsilon \phi(\bar{x})$.

Furthermore, if ϕ is lsc at each point of X, then it is said to be lsc in X; and if ϕ is usc at each point of X with $\phi(x)$ compact for each x, then ϕ is said to be usc in X. Using these definitions, the following results are easily established for real-valued functions (cf. Berge [13]). Kummer [74] has obtained similar results and has applied them to the particular case of quasi-convex programs.

Theorem 2.3. If f is a real-valued usc (lsc) function defined on X x Y and if ϕ is a lsc (usc) mapping from X into Y such that for each x, $\phi(x) \neq \emptyset$, then the (real-valued) function f*, defined by

$$f^*(x) = \inf_{y \epsilon Y} \{f(x,y)\}$$

is usc (lsc).

Theorem 2.4. If f is a continuous real-valued function defined on the space Y and ϕ is a continuous mapping of X into Y such that $\phi(x) \neq \emptyset$. for each $x \epsilon X$, then the (real-valued) function f*, defined by

$$f^*(x) = \min_{y \epsilon \phi(x)} \{f(y)\}$$

is continuous in X. Furthermore, the mapping F, defined by

$$F(x) = \left\{ y : y \epsilon \phi(x), \ f(y) = f^*(x) \right\},$$

is a usc mapping of X into Y.

Conditions which imply the continuity of the solution of a mathematical program have been given by Dantzig, Folkman, and Shapiro [25], and by Robinson and Day [102]. Letting

f be a mapping from a metric space X to E^1, with $H(\epsilon) \subset X$, and defining $M(f:H(\epsilon)) = \left\{ x \epsilon H(\epsilon): f(x) = \inf \left\{ f(y): y \epsilon H(\epsilon) \right\} \right\}$ Dantzig, et. al. obtain conditions on the variation of f and H which are necessary and sufficient for $M(f:H(\epsilon))$ to vary in a closed manner. When H is defined by linear inequalities, they obtain under appropriate conditions, that $M(f:H(\epsilon))$ is a closed function. Under this same hypothesis, letting $M^*(f:H(\epsilon))$ denote the mapping $M(f:h(\epsilon))$ when it is a singleton, they obtain conditions which yield the continuity of $M^*(f:H(\epsilon))$ as a function of the parameter ϵ.

Theorem 2.5. Let H be a point-to-set mapping from the metric space T to the set of subsets of E^n, with $H(\epsilon)$ a closed set for each ϵ in T. Let $T' = \{\epsilon$ in $T: H(\epsilon) \neq \emptyset\}$. Suppose $H(\epsilon)$ is connected for each ϵ in T', and for some ϵ^* in T', $H(\epsilon^*)$ is compact. Furthermore, assume that for every sequence $\{\epsilon_n\} \subset T'$, $\epsilon_n \to \epsilon^*$ implies $\lim_{n \to \infty} H(\epsilon_n) = H(\epsilon^*)$. Then, if $f \epsilon C(E^n)$, the mapping $\epsilon \to M^*(f:H(\epsilon))$

is continuous at ϵ^* if ϵ^* is in its domain.

If g is an affine function from E^n to E^m, denote by $H(g)$ the set $\{x \epsilon E^n: g(x) \geq 0\}$. The function g is said to be nondegenerate with respect to the set $R \subset E^n$ if $H(g) \cap R$ has a nonempty interior and no component of g is identically zero. The continuity of $M(f:H(g) \cap R)$ and $M^*(f:H(g) \cap R)$ as functions of g are given by the following theorems.

Theorem 2.6. If f is continuous and R is closed and convex, then $M(f:H(g) \cap R)$ is closed at every nondegenerate point g.

Theorem 2.7. If f is quasiconvex or $H(g) \cap R$ is bounded, then $M^*(f:H(g) \cap R)$ is continuous at a nondegenerate point g of its domain.

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Robinson and Day [102], considering a general constraint set, $R(\epsilon)$, provide conditions which guarantee the continuity of the point-to-set mapping whose range is the set of solutions of the mathematical program

$$\text{minimize } f(x,\epsilon)$$
$$\text{s.t. } x\epsilon R(\epsilon)$$
$$\epsilon \text{ in } T,$$

where $R(\epsilon)$ represents a constraint set as a function of the parameter ϵ, and X and T are topological spaces. To that end, let $f^*(\epsilon) = \min_x \{f(x,\epsilon): x\epsilon R(\epsilon)\}$, and define the mapping $S:T\rightarrow X$ by $S(\epsilon) = \{x: f(x,\epsilon) \leq f^*(\epsilon)\} \cap R(\epsilon)$. Assume $R(\epsilon)$ is continuous on T.

Theorem 2.8. Let X be locally convex and let $R \neq \emptyset$ be convex-valued on T. If $f(x,\epsilon) = \min\{u(x,\epsilon), \sigma(\epsilon)\}$, where $u\epsilon C(X \times T)$ and is strictly quasi-convex in x for each fixed ϵ, and $\sigma\epsilon C(T)$, then S is continuous and convex valued on T.

2.2 Right-Hand Side (RHS) Perturbations

In this section we will be concerned with problems of the form:

$$\text{minimize } f(x), \text{ s.t., } g(x) \geq \epsilon, \qquad\qquad P_1(\epsilon)$$

where $f: E^n \rightarrow E^1$ and $g: E^n \rightarrow E^m$. Associated with problem $P_1(\epsilon)$ are the following four sets:

   i) the feasible region, $R(\epsilon)=\{x: g(x) \geq \epsilon\}$,

   ii) the set $B =\{\epsilon:R(\epsilon) \neq \emptyset\}$,

   iii) a set associated with the interior of the feasible region, $I(\epsilon) = \{x:g(x) > \epsilon\}$,

   iv) the optimal value function, $f^*(\epsilon) = \inf \{f(x):x\epsilon R(\epsilon)\}$.

One readily sees that i), iii) and iv) may each be viewed as a point-to-set mapping. It is precisely this observation which has been exploited in characterizing the variation of both the solution and the optimal objective function value as functions of the parameter ϵ.

The following three theorems, due to Evans and Gould [34], provide conditions for the stability of the constraint set, i.e. the feasible region $R(\varepsilon)$, as well as for the continuity of the optimal objective function. In the statements of these results, we will denote by R a point-to-set mapping from the set $B \subseteq E^m$ to the set of all subsets of E^n, with the image of ε in B given by $R(\varepsilon)$. The interior of the set B is denoted by $Int(B)$, and the closure of $I(\varepsilon)$ is denoted by $\overline{I(\varepsilon)}$. Assume g is continuous.

Theorem 2.9.   i) The mapping R is usc at ε if and only if there exists a vector $\varepsilon' < \varepsilon$ such that $R(\varepsilon')$ is compact.

ii) If $R(\varepsilon)$ is compact and $I(\varepsilon) \neq \emptyset$, then R is lsc at ε if and only if $\overline{I(\varepsilon)} = R(\varepsilon)$.

Theorem 2.10. i) If f is lsc and R is usc at ε, then f^* is left continuous (lsc) at ε.

ii) If $I(\varepsilon) \neq \emptyset$, f is usc and R is lsc at ε, then f^* is right continuous (usc) at ε.

Theorem 2.11. Suppose ε is in $Int(B)$ and that f is continuous, $f \in C(E^n)$. Also assume that there exists a vector $\varepsilon' < \varepsilon$ such that $R(\varepsilon')$ is compact, and that $\overline{I(\varepsilon)} = R(\varepsilon)$. Then f* is continuous at ε.

Theorem 2.11 is related to the second result of Berge, i.e., Theorem 2.4, in that Evans and Gould have given conditions which imply the hypotheses of Berge's theorem. The question of the stability of the set of optimal solutions and the stability of the optimal objective function value have been addressed by a number of authors. Greenberg and Pierskalla [58], referring to problem P_1, have shown that the point-to-set mapping ϕ, which maps E^m to subsets of E^n and is defined by $\phi(\varepsilon) = \{x \in R(\varepsilon) : f(x) = f^*(\varepsilon)\}$ is usc at ε if R is usc at ε and if f* is continuous at ε. This result is very similar to one given by Dantzig, Folkman, and Shapiro [25] and by Berge [13]. The essential differences among these results lie in the use of semi-continuity in [58] and the closedness of maps in [25], while the conclusion drawn in

[13] is based on the continuity of R.

Considering the perturbation function for convex programming problems, which is central to the construction of decomposition algorithms for large-scale nonlinear programs, Hogan [68] established conditions for its continuity. That result has been extended to the case in which the parameter of the problem appears in the objective function as well as in the constraints. These results are given in the next two theorems.

Theorem 2.12. Let $f^*(\varepsilon) = \inf_x \{f(x): x\varepsilon X, \ g(x) \geq \varepsilon\}$. If

      i) X is a compact convex set,

     ii) f is continuous on X,

    iii) g_i is lsc on X, and

     iv) each g_i is strictly concave on X.

then f^* is continuous on its domain.

Theorem 2.13. Let $f^*(\varepsilon) = \inf_x \{f(x,\varepsilon): x\varepsilon X, \ g(x,\varepsilon) \geq 0\}$. If

      i) X is a compact convex set,

     ii) f and g are both continuous on X, and

    iii) g is strictly concave on X for each ε.

then f^* is continuous on its domain.

More recently, Clarke [21] has shown that if X is a Banach space and f is locally Lipschitz then programs of the form $P_1(\varepsilon)$, with $x\varepsilon S \subset X$ are "normal" in the sense that generalized Kuhn-Tucker conditions can be shown to pertain, even in the absence of differentiability and convexity assumptions. Clarke terms the program

$$\min \ f(x)$$
$$\text{s.t.} \ g \ (x) \geq \ \varepsilon \quad i = 1,\ldots,m,$$
$$x\varepsilon S, \ S \text{ closed in } E^n, \qquad\qquad P_2(\varepsilon)$$

"normal" if Karush-Kuhn-Tucker type multipliers exist for any x which solves $P_2(\varepsilon)$. $P_2(\varepsilon)$ is said to be "calm" if $\phi(\varepsilon)$ is finite and

$$\liminf_{\varepsilon' \to \varepsilon} [\phi(\varepsilon') - \phi(\varepsilon)]/ |\varepsilon' - \varepsilon| > -\infty \qquad (2.1)$$

where

$$\phi(\varepsilon) = \begin{cases} \inf\{f(x): x\varepsilon S, \ g(x) \geq \varepsilon\} & R(\varepsilon) \neq \emptyset \\ \\ +\infty & R(\varepsilon) = \emptyset \end{cases}$$

Note that the limit quotient in (2.1) is a form of stability also used by Rockafellar [105]. Using these notions, Clarke showed that if $P_2(\varepsilon)$ is calm it is also normal and if $\phi(\varepsilon)$ is finite in a neighborhood of $\varepsilon = 0$, then $P_2(\varepsilon)$ is calm and normal for almost all ε in a neighborhood of of 0. Conditions sufficient for the calmness (and hence the normality) of $P_2(\varepsilon)$ are given in the following theorem.

Theorem 2.14. If

     i) $-g_i$, $i = 1,\ldots,m$, are convex,

    ii) S is convex and bounded,

    iii) f is bounded and Lipschitz on S, and

    iv) there exists a point $x\varepsilon S$ such that $g(x) > \varepsilon$,

then $P_2(\varepsilon)$ is calm.

Additional stability results for right hand side perturbations are discussed in the next section where we summarize the results that have been achieved in the area of differential stability. In Section 5, first- and second-order sensitivity of the objective function under right hand side perturbations is indicated.

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3. Differential Stability

In this section we concentrate on theory that has been developed to analyze various differentiability properties of the optimal value function. We begin with a brief discussion of several well known constraint qualifications, one of which is often involved in obtaining some of the results.

3.1 Constraint Qualifications

The constraint qualifications used in mathematical programming are regularity conditions which are generally imposed to insure that the set of Karush-Kuhn-Tucker multipliers corresponding to the optimal solution of a mathematical program is nonempty. We present here five qualifications which are frequently applied. These and a number of others are treated in some detail by Mangasarian [79]. Throughout this discussion, we shall assume the constraint set $R \equiv \{x \epsilon E^n : g(x) \geq 0, h(x) = 0\}$, where $g: E^n \to E^m$ and $h: E^n \to E^p$. Define $B(x) \equiv \{i : g_i(x) = 0\}$.

CQ1. The Mangasarian-Fromovitz constraint qualification is said to hold at a point $x^* \epsilon R$ if:

i)   there exists a vector $y \epsilon E^n$ such that $\nabla g_i(x^*) y > 0$ for all $i \epsilon B(x^*)$, $\nabla h_j(x^*) y = 0$ for $j = 1,\ldots,p$, and

ii)   the gradients $\{\nabla h_j(x^*), j = 1,\ldots p,\}$ are linearly independent.

If the g_i are concave (or even pseudo-concave) functions and the h_j are affine, then CQ1 is equivalent to the well-known Slater condition, a general form of which we give as CQ2.

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<u>Definition 3.1.</u> If the function $f: E^n \to E^1$ is differentiable on the convex set S and $f(y) \geq f(x)$ for all x, y$\in$S, with $\nabla f(x)(y-z) \geq 0$, then f is said to be psuedo-convex on S. (The function f is said to be pseudo-concave if -f is pseudo-convex.)

<u>CQ2.</u>   The Slater constraint qualification is satisfied at x*$\in$R if h_j is affine for each j, g_i is pseudo-concave and there exists a point $\bar{x}\in$R with $g_i(\bar{x}) > 0$ for each i in B(x*).

<u>CQ3.</u>   The linear independence assumption is said to hold at x*$\in$R if the gradients $\{\nabla g_i(x*), i\in B(x*); \nabla h_j(x*), j=1,\ldots,p\}$ are linearly independent.

<u>CQ4.</u> If there are no equality constraints and $\sum_{i\in B(x*)} u_i \nabla g_i(x*) = 0$ has no non-zero solution $u_i \geq 0$ for x*$\in$R, the Cottle constraint qualification is said to hold at x*. (In the absence of equality constraints, CQ1 is equivalent to CQ4.)

<u>CQ5.</u> The Kuhn-Tucker constraint qualification is satisfied at x*$\in$R if, for each non-zero vector z$\in E^n$ satisfying $\nabla g_i(x*)z \geq 0$ for each i$\in$B(x*) and $\nabla h_j(x*)z = 0$, j = 1,\ldots,p, z is tangent to a once-differentiable arc originating at x* and contained in R.

The relationships that hold among these qualifications, in addition to those already mentioned, are that CQ3 implies CQ1 which, in turn, is sufficient for CQ5. For a proof and further discussion of the relationships among these constraint qualifications see [9, 61].

Robinson [101] has shown the equivalence of CQ1 and a form of local stability of the set of solutions of a system of inequalities. Gauvin [49] has shown that CQ1 is both necessary and sufficient for the set of Lagrange multiplier vectors corresponding to a given local

solution of a general NLP problem to be nonempty, compact and convex. In addition, Gauvin and Tolle [50] establish that CQl is preserved under RHS perturbations.

3.2 Differential Stability of the Extremal Value Function

One of the earliest characterizations of the differential stability of the extremal value function of a mathematical program was provided by Danskin [23,24]. Addressing the problem minimize $f(x,\varepsilon)$ subject to $x \in R$, R some topological space, Danskin derived conditions under which the directional derivative of the extremal value function exists and also determined its representation.

Definition 3.2. The (one-sided) directional derivative of the function $f(x)$ in the direction z is defined to be:

$$D_z f(x) = \lim_{\beta \to 0^+} [(f(x+\beta z) - f(x))/\beta]$$

if the limit exists.

Theorem 3.3. Let R be non-empty and compact and let f and the partial derivatives $\partial f/\partial \varepsilon_i$ be continuous. Then, at any point ε in E^k and for any direction $z \in E^k$, the directional derivative of f^* exists and is given by

$$D_z f^*(\varepsilon) = \min_{x \in S(\varepsilon)} z \nabla_\varepsilon f(x,\varepsilon),$$

where $S(\varepsilon) = \{x : x \text{ minimizes } f(x,\varepsilon) \text{ over } R\}$.

This result has wide applicability in the sense that the constraint space, R, can be any compact topological space. It has been extended by a number of authors, including Demyanov and Rubinov [27], to other spaces and a variety of functional forms. The principal restriction of this result is that the set R does not vary with the parameter ε. However, since inequality and equality constraints can be "absorbed" into the objective function of a program through the use of an appropriate auxiliary function (Lagrangian, penalty function, etc.), Danskin's result is readily applicable to auxiliary function methods. It can also be readily applied to the dual of a convex program with right hand side perturbations. For the special case in which R is defined by inequalities, $g(x,\varepsilon) \geq 0$, and f and g are convex on S,

Hogan [66] has shown that $D_z f*(\epsilon)$ exists and is finite for all $z \epsilon E^k$. The following theorem presents the details of this result for $f*(\epsilon) = \inf_{x \epsilon X} \{f(x,\epsilon): g(x,\epsilon) \geq 0\}$, where X is a subset of E^n. The Lagrangian for this problem is defined as $L(x,u,\epsilon) \equiv f(x,\epsilon) - ug(x,\epsilon)$. For convenience and without loss of generality, we shall assume that the parameter value of interest is $\epsilon = 0$, unless otherwise stated.

Theorem 3.4. Let X be a closed and convex set. Suppose f and -g are convex on X for each fixed ϵ and are continuously differentiable on X x N(0), a neighborhood of $\epsilon = 0$ in E^k. If $S(0) \equiv \{x \epsilon X: g(x,0) \geq 0$ and $f*(0) \geq f(x,0)\}$ is nonempty and bounded, $f*(0)$ is finite, and there is a point $y \epsilon X$ such that $g(y,0) > 0$, then $D_z f*(0)$ exists and is finite for all $z \epsilon E^n$, and

$$D_z f*(0)= \min_{x \epsilon S(0)} \max_{u \epsilon K(x,0)} z \nabla_\epsilon L(x,u,\epsilon) = \min_{x \epsilon S(0)} \max_{u \epsilon K(x,0)} \{z(\nabla_\epsilon f(x,0) - u \nabla_\epsilon g(x,0))\},$$

where $K(x,0)$ is the set of optimal Lagrange mulitpliers for the given $x \epsilon S(0)$.

Some recent investigations of this sort have focused on the extremal value function inequality-equality constrained optimization problems with right hand side perturbations, of the form

$$\text{maximize } f(x)$$
$$\text{s.t. } g(x) \geq \epsilon_i \qquad i = 1,\ldots,m$$
$$h(x) = \epsilon_{j+m} \qquad j = 1,\ldots,p. \qquad P_3(\epsilon)$$

Let $R(\epsilon) = \{x: g_i(x) \geq \epsilon_i, i = 1,\ldots,m\} \cap \{x: h_j(x) = \epsilon_{j+m}, j = 1,\ldots,p\}$, and let

$$f*(\epsilon) = \begin{cases} \inf \{f(x): x \epsilon R(\epsilon)\} & R(\epsilon) \neq \emptyset \\ +\infty & R(\epsilon) = \emptyset. \end{cases}$$

We also define, for $R(\epsilon) \neq \emptyset$, $S(\epsilon) = \{x \epsilon R(\epsilon): f(x) = f*(\epsilon)\}$, and the Lagrangian $L(x,u,w,\epsilon) \equiv f(x) - u[g(x)-\epsilon] + w[h(x) -\epsilon]$. Given these definitions, Gauvin and Tolle [50] have proved the following continuity property of $f*(\epsilon)$.

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Theorem 3.5.   If $R(0) \neq \emptyset$, with $R(\varepsilon)$ uniformly compact near $\varepsilon = 0$, and if CQ1 holds for some $x^* \varepsilon S(0)$, then $f^*(\varepsilon)$ is continuous at $\varepsilon = 0$.

Fiacco [45] has recently shown that this result holds for the general problem

$$\begin{aligned} \min \quad & f(x,\varepsilon) \\ \text{s.t.} \quad & g(x,\varepsilon) \geq 0 \\ & h(x,\varepsilon) = 0, \end{aligned} \qquad P_4(\varepsilon)$$

where the problem functions are C^1 in (x,ε), $R(\varepsilon)$ and $S(\varepsilon)$ are defined to be the feasible region and solution sets of $P_4(\varepsilon)$, respectively, and where the parameter ε is a vector in E^k.

In the absence of equality constraints, Rockafellor [103] has shown that, under certain second-order conditions, the function $f^*(\varepsilon)$ of $P_3(\varepsilon)$ satisfies a stability of degree two, i.e., in a neighborhood of $\varepsilon = 0$, there exists a twice differentiable function $\phi: E^m \to E^1$ with $f^*(\varepsilon) \geq \phi(\varepsilon)$ and $f^*(0) = \phi(0)$. Under this stability property, bounds on the directional derivatives of f^* (when they exist) can be derived. For convex programming problems of the form $P_4(\varepsilon)$, Gol'stein [57] has shown that a saddle point condition (summarized subsequently) is satisfied by the directional derivative of f^*. Gauvin and Tolle [50], not assuming convexity, but limiting their analysis to problem $P_3(\varepsilon)$, extend the work of Gol'stein and provide sharp bounds on the directional derivative of f^*, also without requiring the existence of second order derivatives. These results were extended by Fiacco and Hutzler [44] to the more general problem $P_4(\varepsilon)$ and are presented next.

Let $L(x,u,w,\varepsilon) \equiv f(x,g) - ug(x,\varepsilon) + wh(x,\varepsilon)$ denote the usual Lagrangian of Problem $P_4(\varepsilon)$ and let $K(x,0)$ denote the set of Kuhn-Tucker vectors (u,w) corresponding to a solution x of $P_4(\varepsilon)$ at $\varepsilon = 0$, let $Q(z) = (f^*(\beta z) - f^*(0))/\beta$, where $z \varepsilon E^k$ is a unit vector. As above, $R(\varepsilon)$ denotes the feasible region, $S(\varepsilon)$ the solution set, and the functions f, g, h are assumed jointly once continuously differentiable in (x,ε). The following results hold [44] for Problem $P_4(\varepsilon)$.

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<u>Theorem 3.6.</u> If $R(0) \neq \emptyset$, $R(\epsilon)$ is uniformly compact near $\epsilon = 0$, and CQ1 holds for some $x^* \epsilon S(0)$ then, for any $z \epsilon E^k$,

$$\lim_{\beta \to 0^+} \inf Q(z) \geq \min_{(u,w) \epsilon K(x^*,0)} z \nabla_\epsilon L(x^*,u,w,0) \tag{3.1}$$

The next corollary, following immediately from the theorem, gives a result that is weaker, but useful in the sequel.

<u>Corollary 3.7.</u> If $R(0) \neq \emptyset$, $R(\epsilon)$ is uniformly compact near $\epsilon = 0$, and CQ1 holds at each $x \epsilon S(0)$ then, for any $z \epsilon E^k$,

$$\lim_{\beta \to 0^+} \inf Q(z) \geq \inf_{x \epsilon S(0)} \min_{(u,w) \epsilon K(x,0)} z \nabla_\epsilon L(x,u,w,0) \tag{3.2}$$

<u>Theorem 3.8.</u> Under the conditions of Corollary 3.7, for any $z \epsilon E^k$,

$$\lim_{\beta \to 0^+} \sup Q(z) \leq \inf_{x \epsilon S(0)} \max_{(u,w) \epsilon K(x,0)} z \nabla_\epsilon L(x,u,w,0) \tag{3.3}$$

<u>Corollary 3.9.</u> If, in the hypotheses of Corollary 3.7, CQ1 is replaced with CQ3, then for each $z \epsilon E^k$, $D_z f^*(0)$ exists and

$$D_z f^*(0) = \inf_{x \epsilon S(0)} z \nabla_\epsilon L(x,u(x),w(x),0) \tag{3.4}$$

where $(u(x),w(x))$ is the unique optimal Lagrange multiplier vector associated with $x \epsilon S(0)$.

The following theorem obtained by Fiacco and Hutzler [44] corresponds, under slightly different assumptions, to results obtained by Gol'stein [57] and Hogan [66], for a general class of problems that are convex in x.

<u>Corollary 3.10.</u> Let $-f$ and g_i, $i = 1,\ldots,m$, be convex functions in x, and let the functions h_j, $j = 1,\ldots p$, be affine in x, with all functions jointly C^1 in (x,ϵ). If $R(0) \neq \emptyset$, $R(\epsilon)$ is uniformly compact near $\epsilon = 0$, and CQ1 is satisfied for each $x \epsilon S(0)$, then $D_z f^*(0)$ exists for each $z \epsilon E^k$, and

$$D_z f^*(0) = \inf_{x \epsilon S(0)} \max_{(u,w) \epsilon K(x,0)} z \nabla_\epsilon L(x,u,w,0) \tag{3.5}$$

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Lempio and Maurer [77] have obtained similar bounds under analogous assumptions that are required to handle general perturbed infinite dimensional programs of the form minimize $f(x,\varepsilon)$, subject to $x \varepsilon R_1$ and $g(x,\varepsilon) \varepsilon R_2$, where R_1 and R_2 are arbitrary closed convex sets. Auslender [10] has also obtained these bounds for the right-hand side pertubation problem $P_3(\varepsilon)$, extending the results of Gauvin and Tolle [50] by using a weaker form of the Mangasarian-Fromovitz constraint qualification. This allows him to replace the differentiability assumption on the objective and inequality functions with the weaker requirement that they be locally Lipschitz.

Although we are focusing attention on programs for which the spaces involved are finite dimensional, we note that most of these sensitivity results have been extended to infinite dimensional programs. For example, Maurer [82,83] has recently obtained a characterization of the directional derivative of the extremal value function subgradient for problem $P_4(\varepsilon)$, and has applied his results to a class of optimal control problems.

3.3 Lipschitz Properties

In this section, we will consider problem $P_1(\varepsilon)$. Following the notation used in Section 2, recall that $R(\varepsilon) \equiv \{x \varepsilon E^n: g(x) \geq \varepsilon\}$, $B \equiv \{\varepsilon \text{ in } E^k: R(\varepsilon) \neq \emptyset\}$, and $f^*(\varepsilon) \equiv \inf \{f(x): x \varepsilon R(\varepsilon)\}$. In addition, let $S(\varepsilon,\delta) \equiv \{x \varepsilon R(\varepsilon): f(x) \leq f^*(\varepsilon) + \delta \text{ for } \delta > 0\}$.

Stern and Topkis [111], defining a notion of linear continuity, establish conditions under which $f^*(\varepsilon)$ satisfies a Lipschitz condition. Under convexity assumptions on the problem functions, they also show that $S(\varepsilon,\delta)$, the set of δ-optimal solutions, is continuous.

Definition 3.11. The real-valued function $f(x)$ is said to satisfy a Lipschitz condition on a set S if there exists a value $M > 0$ such that $|f(x)-f(y)| \leq M \cdot ||x-y||$, for all $x, y \varepsilon S$.

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<u>Definition 3.12.</u> Suppose ϕ is a point-to-set mapping from $Y \subseteq E^m$ to subsets of E^n. Then ϕ is said to be uniformly linearly continuous on $\bar{Y} \subseteq Y$ if there exists a value $M > 0$ such that $\inf_{x \varepsilon \phi(y)} ||z-x|| \leq M \cdot ||y-\bar{y}||$, for all $z \varepsilon \phi(y)$ and for all $y, \bar{y} \varepsilon \bar{Y}$.

<u>Theorem 3.13.</u> Let $\bar{B} \subseteq B$ and suppose $R(\bar{\varepsilon})$ is bounded for $\bar{\varepsilon}$ in B. If $R(\varepsilon)$ is uniformly linearly continuous with constant K on $\bar{B} \cap \{\varepsilon : \varepsilon \leq \bar{\varepsilon}\}$, and if f satisfies a Lipschitz condition with constant M on $R(\bar{\varepsilon})$, then $f^*(\varepsilon)$ satisfies a Lipschitz condition with constant KM on $\bar{B} \cap \{\varepsilon : \varepsilon \leq \bar{\varepsilon}\}$.

<u>Theorem 3.14.</u> If f, and $-g_i$, $i = 1, \ldots, m$, are strictly quasi-convex, $R(\bar{\varepsilon})$ is bounded, and $I(\bar{\varepsilon}) \neq \emptyset$, then $S(\varepsilon, \delta)$ is continuous at $\bar{\varepsilon}$ for each $\delta > 0$.

Hager [61] has recently obtained Lipschitz results for quadratic programs with unique solutions. Hager obtained the Lipschitz continuity of the solution of such programs under the hypothesis that the gradients of the binding constraints satisfy an independence criterion. Under these same conditions, the Lipschitz continuity of the (unique) Lagrange multiplier vector is also obtained. In both instances, Hager provides an estimate of the Lipschitz constant.

4. Implicit Function Theorems

There are many forms of implicit function theorems which have found extensive application in functional analysis. These theorems treat the general problem of solving an equation of the form

$$f(x,y) = z \qquad (4.1)$$

for x in terms of y and z. The classical results in this area are by now well known. For completeness, we present two forms of the implicit function theorem. A more complete discussion of these theorems is contained in [16].

Theorem 4.1. Suppose $f: E^{n+m} \to E^n$ is a continuously differentiable mapping whose domain is T. Suppose $(\bar{x},\bar{y}) \in T$, $f(\bar{x},\bar{y}) = 0$, and $f'(x,0) = 0$ implies $x = 0$. Then there exists a neighborhood of $\bar{y}$, $N(\bar{y}) \subseteq E^m$, and a unique function $g \in C^1(N(\bar{y}))$, $g:N(\bar{y}) \to E^n$, with $g(\bar{y}) = \bar{x}$ and $f(g(y),y) = 0$ for all $y \in N(\bar{y})$.

The function g is said to be defined implicitly by the equation $f(g(y),y) = 0$. In the next theorem, the notation f_j is used to denote the partial derivative of f with respect to its jth argument.

Theorem 4.2. If $f_j(x_1,\ldots,x_k;y_1,\ldots,y_\ell)$ is analytic in a neighborhood of the origin for $j = 1,\ldots,k$, with $f(0,0) = 0$ and $[\partial(f_1,\ldots,f_k)/\partial(x_1,\ldots x_k)]^{-1}$ exists at $x = y = 0$, then the system of equations $f_j(x_1,\ldots,x_k;y_1,\ldots y_\ell) = 0$, for $j = 1,\ldots,k$, has a unique solution $x_j = x_j(y_1,\ldots,y_\ell)$, which vanishes for $y = 0$ and which is analytic in a neighborhood of the origin.

Results of this type have particular applicability to sensitivity analysis in nonlinear optimization and have only recently been exploited. Hildebrandt and Graves [63] have provided results on the existence and differentiability of solutions of equation (4.1). Cesari

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[20] has established conditions under which the equation $f(y,y) = 0$ has at least one solution, and discusses the continuous dependence of y on parameters of the equation. Rheinboldt [93] has given global existence theorems for the solution of (4.1) which leads to a "continuation property". This continuation property has been applied to the solution of parametric optimization programs.

Fiacco and McCormick [45] provided a first application of an implicit function theorem to obtaining sensitivity information about the solution of a mathematical program. Since then, additional results in this area have been obtained by Bigelow and Shapiro [14], Armacost and Fiacco [3,4,5,6], Armacost [1], Fiacco [41], and Robinson [98].

In [94], Robinson provided a implicit function theorem for variational inequalities of the form

$$0 \ \varepsilon \ f(x,\varepsilon) + T(x), \tag{4.2}$$

where $f:X \times P \ E^n$, X is an open set in E^n and P is a topological space, and $T:E^n \rightarrow E^n$ is a maximal monitone operator. Robinson showed that if f is continuously (Frechet) differentiable on $X \times P$, and $X_o \subseteq X$ is nonempty and bounded, then the set of solutions $S(\varepsilon)$ of (4.2) is u.s.c. in a neighborhood of $\varepsilon = \bar{\varepsilon}$; $S(\varepsilon) = X_o$; and for each $\delta > 0$,

$$\emptyset \neq S(\varepsilon) \subset S(\varepsilon) + (\lambda + \delta) \ \alpha(\varepsilon) \ B, \tag{4.3}$$

where

$$\alpha(\varepsilon) = \max \ \{||f(x,\varepsilon) - f(x,0)||: \ x \varepsilon X_o\},$$

B is the unit sphere in E^n, and λ is a Lipschitz modulus regulating $(Lf_o + T)^{-1}$. Here,

$$Lf_o(\cdot) = f(x_o,\varepsilon) + F(x_o,\varepsilon) \ [(\cdot) - x_o]$$

where F, the Frechet derivative of f, is positive semi-definite and $(Lf_o + T)^{-1} (0) = X_o$.

As Robinson indicates, nonlinear complementarity problems, and thus the Kuhn-Tucker conditions for mathematical programming, can be written

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in the form of (4.2). Robinson's result (see [95]) when applied to linear equations of the form

$$0 \in Ax + a + T(x),$$ (4.4)

where A is an n x n matrix, $a \in E^n$, and T(x) is the subdifferential operator of the indicator function of a nonempty polyhedral convex set, leads to the relation

$$\emptyset \neq S(A',a') \cap X \subset S(A,a) + \lambda \varepsilon'(1-\lambda \varepsilon')^{-1}(1+u)B.$$ (4.5)

Here S(A,a) is the set of solutions of (4.4), X is any bounded open set containing S(A,a), $\varepsilon' = \max \{||A'-A||, ||a'-a||\}$, μ is a bound on S(A,a) and λ is a Lipschitz modulus for $[A + a + T]^{-1}$.

These results can be applied directly to quadratic programs, and in that context (4.5) can be viewed as an extension of Daniel's [22] result on the solution stability for definite quadratic programs, which although it does not involve an implicit function theorem, is given next for comparison with (4.5). Daniel considers the program

$$\min \quad \tfrac{1}{2}(x'Kx) - xk$$
$$\text{s.t.} \quad Gx \leq g$$
$$\quad Dx = d,$$

where K is positive definite and symmetric with $\tilde{\lambda}>0$ its smallest eigenvalue. Letting $|| \cdot ||$ be the ℓ_2 norm, Daniel obtained the following special case of (4.5).

Theorem 4.3. If $\varepsilon = \max\{||K'-K||,||k'-k||\}$, then for $\varepsilon > \tilde{\lambda}$,
$$||x_0' - x_0|| \leq \varepsilon (\tilde{\lambda}-\varepsilon)^{-1}(1 + ||x_0||),$$

where x_0 solves the program above and x_0' solves that program when K' and k' replace K and k, respectively.

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5. First- and Second-Order Sensitivity Analysis

Using additional assumptions, a number of stronger results have been obtained which characterize more completely the relationship between a solution set and the optimal value function of a mathematical program to general perturbations appearing simultaneously in the objective function and anywhere in the constraints. These problems generally have the form $P_4(\varepsilon)$, which we treated briefly in Section 3 and we formulated as follows:

$$
\begin{aligned}
&\text{minimize } f(x,\varepsilon) \\
&\text{s.t. } g(x,\varepsilon) \geq 0 \qquad\qquad\qquad P_4(\varepsilon)\\
&\qquad\ h(x,\varepsilon) = 0,
\end{aligned}
$$

where: $f:E^n \times E^k \rightarrow E^1$, $g:E^n \times E^k \rightarrow E^m$, and $h:E^n \times E^k \rightarrow E^p$.

Fiacco and McCormick [45] have obtained conditions which guarantee the existence of a differentiable function of ε which locally solves a particular form of $P_4(\varepsilon)$. Fiacco [41], Armacost and Fiacco [3,4,5,6], and Robinson [98] have extended this result to programs in which the pertubations appear as in $P_4(\varepsilon)$. All of these results rely on a form of the implicit function theorem in order to establish the existence of a differentiable solution of $P_4(\varepsilon)$. The next theorem, due to Fiacco [41], establishes the existence of a once continuously differentiable (local) solution of $P_4(\varepsilon)$.

Theorem 5.1. If

F1) f, g, h are C^2 in (x,ε) in a neighborhood of $(x*,0)$,

F2) the second-order sufficiency conditions hold at $[x*,u*,w*]$,

F3) the linear independence (LI) assumption holds at $x*$, and

F4) $u_i^* > 0$ for all i such that $g_i(x*) = 0$, i.e., strict complementary slackness (SCS) with respect to u^* holds at $x*$.

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then

> i) x* is a local isolated minimizing point of $P_4(0)$ with unique Lagrange multipliers u*, w*,
>
> ii) for ε near 0, there exists a unique C^1 function $y(\varepsilon) = [x(\varepsilon), u(\varepsilon), w(\varepsilon)]$ satisfying the second-order sufficiency conditions for problem $P_4(\varepsilon)$, with $y(0) = [x^*, u^*, w^*]$, and
>
> iii) for ε near 0, the gradients of the binding constraints are linearly independent, and strict complementary slackness holds for $u(\varepsilon)$ and $g(x,\varepsilon)$.

Jittorntrum [70] has obtained results that do not require the strict complementarity condition (F4). Under these weakened assumptions, (i) and (ii) (less the differentiability of $y(\varepsilon)$) pertain, along with the local differentiability of $f^*(\varepsilon) = f(x(\varepsilon), \varepsilon)$ at $\varepsilon = 0$. In addition, the triple $(x(\varepsilon), u(\varepsilon), w(\varepsilon))$ is shown to satisfy a Lipschitz condition in each of its components at $\varepsilon = 0$.

Spingarn [110] has also extended the above result by considering the problem $P_4(\varepsilon)$ with additional constraints that restrict ε to a C^2 submanifold, P, in E^k and restrict x to a "cyrtohedron" of class C^2 in E^n. He has shown a certain set of second-order conditions to be necessary for optimality, and that these conditions also imply the results obtained by Fiacco [41]. Before stating this result, we must introduce additional notation and define several terms. This notation follows that found in [110].

Let A and B be finite (possibly empty) index sets, and for $i \varepsilon A$ and $j \varepsilon B$, let $\{g_i\}$ and $\{h_j\}$ be finite collections of C^1 functions defined and the open set $U \subset E$. Also, for $x \varepsilon U$ and $A' \subset A$, let

$$\Gamma(x, A') = \{\nabla g_i(x) : i \varepsilon A'\} \cup \{\nabla h_j(x) : j \varepsilon B\}, \text{ and}$$

$$Z(A') = \{x \varepsilon U : g_i(x) = 0 = h_j(x) \text{ for all } i \varepsilon A' \text{ and } j \varepsilon B\}.$$

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<u>Definition 5.2</u>. Let S be a nonempty connected subset of E^n. Then, for $K \geq 1$, S is a cyrtohedron of class C^k if there exist sets of C^k functions $\{g_i\}$ for $i \varepsilon A$ and $\{h_j\}$ for $j \varepsilon B$, defined on a neighborhood N of $x^* \varepsilon E^n$ with:

  i) $x^* \varepsilon Z(A)$, and for $x \varepsilon N$, $x \varepsilon S$ if and only if $g_i(x) \geq 0$
     for all $i \varepsilon A$ and $h_j(x) = 0$ for all $j \varepsilon B$,

  ii) if $-\sum a_i \nabla g_i + \sum b_j \nabla h_i = 0$ for some a_i, b_j with $a_i \geq 0$, then
      $a_i = b_j = 0$ for all i, j, and

  iii) $A \subset A_1 \subset A$ and $\Gamma(x^*, A_1) \subset$ span $\Gamma(x^*, A_0)$ implies that
       $Z(A_0) = Z(A_1)$.

Consider now the problem $P_4'(\varepsilon)$ which is $P_4(\varepsilon)$ with the additional constraints mentioned above. The following definition contains conditions which are sufficient for optimality in $P_4'(\varepsilon)$.

<u>Definition 5.3</u>. Let S be a cyrtohedron of class C^2. The point $y^* = (x^*, u^*, w^*)$ is said to satisfy the strong second-order conditions for $P_4'(\varepsilon)$ if:

  i) $x^* \varepsilon \{x : g(x, \varepsilon) > 0\} \cap \{x : h(x, \varepsilon) = 0\}$,

  ii) $- \nabla L(x^*, u^*, w^*, \varepsilon)$ is the relative interior of the
      normal cone to S at x^*,

  iii) the gradients of the constraints that are binding at
       x^* are linearly independent,

  iv) for each $i = 1, \ldots, m$, $u_i^* > 0$ if and only if $g_i(x^*, \varepsilon) = 0$,
      and

  v) $z'[\nabla^2 L(x^*, u^*, w^*, \varepsilon) + K(\nabla L(x^*, u^*, w^*, \varepsilon))]z > 0$ for all
     non-zero $z \varepsilon E^n$ for which

     a) z is in the largest linear subspace contained in
        the tangent cone to S at x^*,

     b) $z' \nabla g_i(x^*, \varepsilon) = 0$ for all $i \varepsilon B^*(\varepsilon)$, and

     c) $z' \nabla h_j(x^*, \varepsilon) = 0$ for $j = 1, \ldots p$,

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where $K(\cdot)$, the curvature of the facial submanifold of S which contains x*, is an n x n matrix.

If the set S is taken to be E^n, then $P_4'(\varepsilon)$ reduces to the program $P_4(\varepsilon)$ and (ii) and (v) above become the familiar conditions

ii') $\nabla L(x^*, u^*, w^*, \varepsilon) = 0$, and

v') $z' \nabla^2 L(x^*, u^*, w^*, \varepsilon) z > 0$ for all non-zero $z \varepsilon E^n$ for which
(b) and (c) above hold.

With this background, we now state Spingarn's result.

Theorem 5.4.   Consider the problem $P_4'(\varepsilon)$. If the strong second-order conditions hold at $y^* = (x^*, u^*, w^*) \varepsilon S \times E^m \times E^p$, ε^* in E^k, then there exist neighborhoods $N \subset E^k$ and $X \subset E^n$ of ε^* and x* respectively, and a C^1 function $y(\varepsilon) = (x(\varepsilon), u(\varepsilon), w(\varepsilon))$ defined on N such that:

   i) $y(\varepsilon)$ satisfies the strong second-order conditions for $P_4'(\varepsilon)$,

   ii) for each ε in N, $x(\varepsilon)$ is an isolated local minimizer for $P_4'(\varepsilon)$, and

   iii) for each ε in N, the Lagrange multipliers $u(\varepsilon)$, $w(\varepsilon)$, associated with $x(\varepsilon)$ are uniquely determined.

Under slightly weaker assumptions than those invoked by Fiacco [41], Robinson [98] has obtained results similar to those stated in Theorem 5.1, proving the continuity of the Kuhn-Tucker triple, and using the results to derive bounds on the variation of $y(\varepsilon)$.

Theorem 5.5.   Let B be a Banach space, $S \subset B$, $X \subset E^n$, with X and S open sets. Let f, g, and h have second partial derivatives with respect to x which are jointly continuous on X x S. For ε^* in S, suppose (x^*, u^*, w^*) is a Kuhn-Tucker triple of $P_4(\varepsilon)$. Also assume that LI, SCS, and the second-order sufficiency conditions apply at (x^*, u^*, w^*). Then

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i) there exists a continuous function $y(\varepsilon)$ with
$y(\varepsilon^*) = (x^*, u^*, w^*)$, and for each ε in S, $y(\varepsilon)$ is the unique
Kuhn-Tucker triple of $'P_4(\varepsilon)$ and the unique zero of
$[\nabla L(x,u,w,\varepsilon), u_1 g_1(x,\varepsilon),\ldots,u_m g_m(x,\varepsilon), h_1(x,\varepsilon),\ldots,h_p(x,\varepsilon)]$,

ii) for ε near ε^*, $x(\varepsilon)$ is an isolated local minimizing
point of $P_4(\varepsilon)$, and

iii) LI, SCS, and the second-order sufficiency conditions
hold for ε near ε^*.

<u>Theorem 5.6.</u> Under the hypotheses of the previous theorem, for any
$\lambda \varepsilon (0,1)$, there exist neighborhoods N_λ^1 of ε^* and N_λ^2 of (x^*, u^*, w^*) such that
for any ε in N_λ^1 and any y in N_λ^2 we have:
$$||y - y(\varepsilon^*)|| \leq (1-\lambda)^{-1} || M(y(\varepsilon^*),\varepsilon^*)^{-1}|| \cdot ||G(y,\varepsilon)||,$$
where M is the Jacobian of $G = [\nabla L(x,u,w,\varepsilon), u_1 g_1,\ldots,u_m g_m, h_1,\ldots,h_p]$.

If instead of the Lagrangian of $P_4(\varepsilon)$ we consider the logarithmic-
quadratic penalty function:
$$W(x,\varepsilon,r) = f(x,\varepsilon) - r \sum_{i=1}^{n} \ln (g(x,\varepsilon) + 1/(2r) \sum_{j=1}^{p} h(x,\varepsilon),$$
we have the following theorem due to Fiacco [34].

<u>Theorem 5.7.</u> Assume (F1)-(F4) above. Then, in a neighborhood of
$(\varepsilon,r) = (0,0)$ there exists a unique, once continuously differentiable
function $y(\varepsilon,r) = [x(\varepsilon,r),u(\varepsilon,r),w(\varepsilon,r)]$ satisfying:

(KT1) $\nabla L(x,u,w,\varepsilon) = 0$

(KT2) $u_i g_i(x,\varepsilon) = r$ $i = 1,\ldots,m$, and

(KT3) $h_j(x,\varepsilon) = w_j r$ $j = 1,\ldots,p$,

with $y(0,0) = [x^*,u^*,w^*]$. Furthermore, for any (ε,r) near $(0,0)$ with
$r > 0$, $x(\varepsilon,r)$ is a locally unique unconstrained local minimizing point

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of $W(x,\varepsilon,r)$ with $g_i(x(\varepsilon,r),\varepsilon) > 0$ for each $i = 1,\ldots m$, and $\nabla^2 W(x(\varepsilon,r),\varepsilon)$ positive definite.

Using a number of the results stated above, Fiacco [41] has obtained first-order (Taylor) approximations of $y(\varepsilon)$. The existence of higher order derivatives of $y(\varepsilon)$ is easily seen to depend on the degree of (continuous) differentiability of the problem functions. This follows directly from an application of the implicit function theorem (cf. [13]). An analogous result holds for $y(\varepsilon,r)$. In fact, Fiacco [41] has shown that, under the appropriate conditions, not only do higher order derivatives of $y(\varepsilon,r)$ exist, but these derivatives converge to the corresponding derivatives of $y(\varepsilon)$.

Theorem 5.8. Let f, g, and h have continuous derivatives of all orders up to k+1. Assume that (F2)-(F4) apply. Then, in a neighborhood of $(\varepsilon,r) = (0,0)$, there exists a unique function $y(\varepsilon,r)\varepsilon\ C^k$, $y(\varepsilon,r) = [x(\varepsilon,r),u(\varepsilon,r),w(\varepsilon,r)]$ satisfying (KT1)-(KT3), with

$$y(\varepsilon,r) \to y(\varepsilon), \text{ and}$$
$$[d^j/d\varepsilon^j]y(\varepsilon,r) \to [d^j/d\varepsilon^j]y(\varepsilon) \quad j = 1,\ldots,k,$$

as $r \to 0$ for (ε,r) near $(0,0)$.

Armacost and Fiacco [2] have illustrated computational aspects of the convergence properties outlined above. Using the SUMT (Sequential Unconstrained Minimization Technique) computer code developed by Mylander, Holmes, and McCormick [91], and a subroutine for sensitivity analysis coded by Armacost and Mylander [7] that implements a procedure based on the results given in Theorems 5.12 and 5.13, they demonstrated the convergence of the first partial derivatives of the optimal solution and the optimal value function of several problems. Subsequently, Armacost and Fiacco [6] used this computer program to analyze the behavior of the solution of an inventory problem relative to changes in several problem parameters.

Armacost and Fiacco [6] have established the first- and second-order changes in the optimal objective function of the problem $P_4(\varepsilon)$ by way of the following theorem.

Theorem 5.9. Assume that (F1)-(F4) hold for $P_4(\epsilon)$. Then, in a neighborhood of $\epsilon = 0$, the extremal value function $f^*(\epsilon)$ is twice continuously differentiable as a function of ϵ, and

$$i)\ f^*(\epsilon) = L^*(\epsilon),$$
$$ii)\ \nabla_\epsilon f^*(\epsilon) = \nabla_\epsilon L(x(\epsilon),u(\epsilon),w(\epsilon),\epsilon),\ \text{and}$$
$$iii)\ \nabla_\epsilon^2 f^*(\epsilon) = \nabla_\epsilon[\nabla_\epsilon L(x(\epsilon),u(\epsilon),w(\epsilon),\epsilon)\ ].$$

This sensitivity result, obtained for the usual Lagrangian, has been extended by Armacost and Fiacco [3] to the augmented Lagrangian used by Buys [18] and Buys and Gonin [19] to obtain related sensitivity results. The augmented Lagrangian is defined as:

$$\phi(x,u,w,\epsilon,c) = f(x,\epsilon) - \sum_{i\in J}(u_i - (1/2)cg_i(x,\epsilon))g(x,\epsilon)$$
$$+ \sum_{j=1}^{p}(w_j + (1/2)ch_j(x,\epsilon))h_j(x,\epsilon) - (1/2c)\sum_{i\in k} u_i^2,$$

where $J = \{i:u_i - cg_i(x,0) \geq 0\}$ and $K = \{i:u_i - cg_i(x,0) < 0\}$.

Theorem 5.10. Under the assumptions (F1)-(F4), for ϵ near 0 and $c > c^*$, there exists a unique C^1 function $y(\epsilon,c) = [x(\epsilon,c),u(\epsilon,c),w(\epsilon,c),u(\epsilon,c)]$ satisfying:

$$i)\ \nabla\phi(x,u,w,\epsilon,c) = 0$$
$$ii)\ u_i g_i(x,\epsilon) = 0 \quad i = 1,\ldots,m,\ \text{and}$$
$$iii)\ h_j(x,\epsilon) = 0 \quad j = 1,\ldots,p,$$

with $[x(\epsilon,c),u(\epsilon,c),w(\epsilon,c)] = y(\epsilon)$. Furthermore, for any ϵ near 0 and $c > c^*$ we have that $x(\epsilon,c)$ is a locally unique unconstrained local minimizing point of $\phi[x,u(\epsilon,c),w(\epsilon,c),\epsilon,c]$ and $\nabla^2\phi$ is positive definite for $[x,u,w]$ near $[x^*,u^*,w^*]$.

Armacost and Fiacco [5] have also obtained first- and second-order expressions for changes in the extremal value function as a function of right hand side perturbations. Consider the problem

$$\text{minimize } f(x)$$
$$\text{s.t. } g(x) > \epsilon_i \quad i = 1,\ldots,m \qquad P_3(\epsilon)$$
$$h(x) = \epsilon_{j+m} \quad j = 1,\ldots,p$$

The Lagrangian for $P_3(\epsilon)$ is given by:

$$L(x,u,w,\epsilon) = f(x) - \sum_{i=1}^{m} u_i [g_i(x) - \epsilon_i] + \sum_{j=1}^{p} w_j [h_j(x) - \epsilon_{j+m}]$$

The following theorem applies to this construct.

<u>Theorem 5.11.</u> Let f, g, and h be twice continuously differentiable in x in a neighborhood of x^* for ϵ near 0. Assume (F2)-(F4) apply to $P_3(\epsilon)$. Then, in a neighborhood of $\epsilon = 0$,

$$\text{i)} \quad \nabla_\epsilon f^*(\epsilon) = [u(\epsilon), -w(\epsilon)] \text{ , and}$$

$$\text{ii)} \quad \nabla_\epsilon^2 f^*(\epsilon) = [\nabla_\epsilon u(\epsilon), -\nabla_\epsilon w(\epsilon)].$$

6. Additional Results, Applications and Future Research

We have endeavored to present a number of basic contributions to the
theory of sensitivity and stability analysis for general classes of nonlinear
programming problems. Hopefully, we have captured the main thrust and range
of developments for the static nonconvex deterministic problem that
relate the behavior of the optimal value or solution set to perturbations
of problem parameters. We have in this brief account omitted a number of
interesting results and implications that weave the fabric more tightly,
and we have not covered results that significantly exploit additional
problem structure. In particular, there are numerous results obtained
by Rockafellar in his book [104] and in several papers that utilize
convexity and duality properties to characterize stability. Geoffrion
[52], Gol'stein [57], Hogan [68] and others have also contributed further
basic results in convex programming. Quadratic programming stability
characterizations have been rather thoroughly developed by Boot [17],
Guddat [60], and Daniel [22], while the parametric range analysis techniques
for linear programming are now well-known and routinely implemented, with
the stability characterizations being further extended by Mills [88],
Williams [113], Dantzig, Folkman and Shapiro [25], Bereanu [12], Martin
[81], and Robinson [95,100]. Further results involving the exploitation
of other structures, for example by Dembo [26] for geometric programming,
have recently been obtained, and others, e.g., for separable [36] and
factorable programming [85] should be forthcoming.

It should also be emphasized that more general treatments of parametric
stability results have been obtained recently, primarily by Robinson
[94-101], who has provided both a basic theoretical framework and a
number of deep stability characterizations. Their generality encompasses
applications to complimentarity and equlibrium problems, as well as to
mathematical programming problems, both in finite and infinite dimensional
spaces, and involve effective use of monotone operators, convex analysis
and contemporary techniques. We should also mention the many general
results obtained by Kummer [74] which synthesize a good bit of the theory
utilizing the point-to-set mapping constructs.

We should also mention that an entire body of theory, somewhat more general but to a great extent anologous to the theory involving parametric variation, has developed more or less in parallel with the parametric theory. This involves the study of the effects of general perturbations of the problem functions on solution behavior. For example, consider P_k: $\min_x f^k(x)$ s.t. $g^k(x) \geq 0$, $h^k(x) = 0$, where the functions f^k, g^k, h^k converge in some specified sense to f, g, h, respectively, as $k \to \infty$. Questions concerning the relationship of solutions of P_k to solutions of P: $\min_x f(x)$ s.t. $g(x) \geq 0$, $h(x) = 0$, are of interest and obviously relate to sensitivity and stability questions. Numerous references could be given here, in addition to those already provided for the parametric perturbations which are certainly relevant (noting that the problem $P(\varepsilon)$ may be analyzed at $\varepsilon = \bar{\varepsilon}$ by considering problems of the form P_k where $f^k(x) = f(x, \varepsilon_k)$, etc., where $\varepsilon_k \to \varepsilon$ as $k \to \infty$). The interested reader is referred to the recent work of Salinetti and Wets [106] that gives a number of interesting results involving sequences of convex sets and their application to convex stochastic programming as well as many references to other work in this area. This application reminds us that the general area of stochastic programming has not been addressed in this survey, either, although the inevitable presence of uncertainty, e.g. parameters that are random variables, would obviously suggest that perturbation analysis results characterizing solution sensitivity or stability would clearly be applicable. Explicit connections have already been made, as suggested by Salinetti and Wets [106] and also by Bereanu [12].

Sensitivity and stability analysis results are ready for extensive computational implementation. Experimental results have unequivocally demonstrated the practical applicability of various computational schemes that can generate a wealth of information that should be extremely valuable to users. The most extensive computations have apparently been performed by Armacost and Fiacco [2,6], utilizing a computer program now called "SENSUMT". This interfaces a subprogram calculating sensitivity information with SUMT [91] and was developed by Armacost and Mylander [7], based on the theory developed by Fiacco [41]. The approach defined and validated in [41] is based on utilizing the

information generated by a solution algorithm to calculate sensitivity
information as a solution is approached. The particular algorithm for
which the theory is developed in detail in [41] is the algorithm based
on the logarithmic-quadratic mixed barrier-penalty function algorithm
for the problem involving inequalities and equalities and general
parametric perturbations. However, the general approach may
conceivably be applied to any algorithm and should yield an efficient
procedure for adding a sensitivity analysis capability to a nonlinear
programming code, once the sensitivity formulas appropriate to the
general algorithmic manipulations and data organizations have been
obtained. Armacost and Fiacco [5] have obtained efficient formulas
for the general problem in terms of the given problem functions
(without presupposing a given solution algorithm is used), and
have also developed formulas for a class of exact penalty functions
[3], the latter also having been obtained by Buys and Gonin [19].

   Armacost [1] has obtained results in terms of an exponential
penalty function and a general "sequential" class of algorithms.
Fiacco [38] has provided results in the context of projected gradient
and reduced gradient calculations. Recent computational experiments
using SENSUMT have been conducted by Fiacco and Ghaemi [42,43], who,
for example, make numerous inferences concerning the solution of
a 22-variable stream-water pollution-abatement model from an analysis
of the sensitivity information deriving from 64 model parameters.
Dinkel and Kochenberger [30,31] and Dinkel, Kochenberger and Wong [32]
have also reported the successful generation of sensitivity
information for several geometric programs based on practical
examples as well as the value of resulting interpretations. Pre-
liminary computational work has thus begun and the practical feasibility
and applicability of generating rather intricate sensitivity information
for nontrivial nonlinear programming problems has been demonstrated.
Widespread implementation and routine use of this capability is now
enthusiastically encouraged and will hopefully not be long in
coming.

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Concerning the recognition by the scientific community of mathematical programmers of the importance and practical ability of generating sensitivity information as part of the usual output of a nonlinear programming code, we note that in a questionnaire, "Survey on Mathematical Programming Software Performance Indicators," circulated by the Mathematical Programming Society in May 1979, two items solicit information regarding the importance attached to the provision of solution sensitivity information. We are happy to see the emergence of interest in this vital requirement, but are disturbed that the interest is so modest and late in arriving, even to sophisticated theoreticians and practitioners. In this context, we also note that the first conference to our knowledge devoted exclusively to sensitivity and stability questions in mathematical programming took place in May 1979 [40].

Another area of research in mathematical programming that is relevant to sensitivity and stability analysis is the development of bounds on the optimal value function value or on the components of an optimal solution vector. These bounds are frequently obtained by generating simpler functions that bound the given problem functions, e.g., convex envelopes of the constraint functions, as in the separable nonconvex programming approach of Falk and Soland [36] or convex underestimating or concave overestimating functions, as in the nonconvex factorable programming approach of McCormick [85]. But once a procedure for generating simpler "bounding problems" is at hand, it can generally be applied to a perturbation of the original problem to obtain "simple" (e.g., convex) bounding problems. If the perturbation analysis of the simpler associated problems is tractable then, once the relationship between the optimal solution of the perturbed bounding problems and the original perturbed problem is understood, we have a procedure for generating bounds on the perturbed solution of the original problem. Geoffrion [53] gives a number of valuable insights and several computationally implementable schemes for obtaining bounds on the variation of the optimal value function of a given problem, in terms of bounds on the variation of the objective function of the given problem, for example. Bounds

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results such as those obtained by Daniel [22] for quadratic programming, a specific instance of which was given in Theorem 4.3, and results at a very high level of generality obtained by Robinson [91] and indicated in Section 4, giving bounds on the optimal solution set in terms essentially of bounds on the original problem functions, are also applicable here. To our knowledge, there has been no theoretical or computational exploitation of the approach suggested in this paragraph, other than that reported by Geoffrion [53]. Development of this idea, including a study of the connection with sensitivity and stability theory of other procedures for generating solution bounds, for example, techniques utilizing the use of interval arithmetic proposed by Robinson [96] and Mancini and McCormick [78], and utilizing many of the results known for systems of equations such as that tailored to the Karush-Kuhn-Tucker conditions by Robinson [98] and stated in Theorem 5.6, should be a subject of fruitful research.

Some applications of sensitivity and stability analysis in mathematical programming are rather obvious, e.g. estimation of solutions of perturbed problems, given a solution of a problem with given parameter values, and determination of parameters to which the optimal value or solution set is most sensitive. Most applications are reasonably well documented. We mention several here for completeness, along with some references: (1) optimality conditions [41,45], (2) convergence of algorithms [87], (3) rate of convergence of algorithms [98], (4) decomposition [54,76], and (5) implicit function minimization [28,65]. Other applications can surely be made to parametric nonlinear programming and deformation techniques [55,56,69, 99], homotopy continuation methods [35,47,85], and to the derivation, conditioning and acceleration of alogrithms [41,45]. Conversely, all of the areas mentioned can undoubtedly uncover an abundance of results that are applicable to sensitivity and stability results in mathematical programming.

The future should see a unification of the powerful collection of results that are now known and scattered throughout the literature. The reader may note that only one book has been devoted entirely

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The future should see a unification of the powerful collection of results that are now known and scattered throughout the literature. The reader may note that only one book has been devoted entirely

to sensitivity analysis in mathematical programming, the book by Dinkelbach [33], and that appeared in 1969 and was confined to linear programming. Unification presupposes the existence of enough interesting results to essentially provide a significant body of theory, a methodology, and this has only been extensively developed in the recent past. Certain efforts to synthesize the theory have begun. We mention the works of Dantzig, Folkman and Shapiro [25], Rockafellar [104], Geoffrion [51], Gol'stein [57], Hogan [67], Robinson [94-101], Gal [46], Fiacco [41], Kummer [74], Wets [112], Dembo [26], Kojima [72], and, of course this survey. A unified methodology will accelerate the understanding of basic theory and stimulate algorithms and software developments, and thus hasten widespread and routine implementations.

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